

# MATH 2050C Lecture 25 (Apr 22)

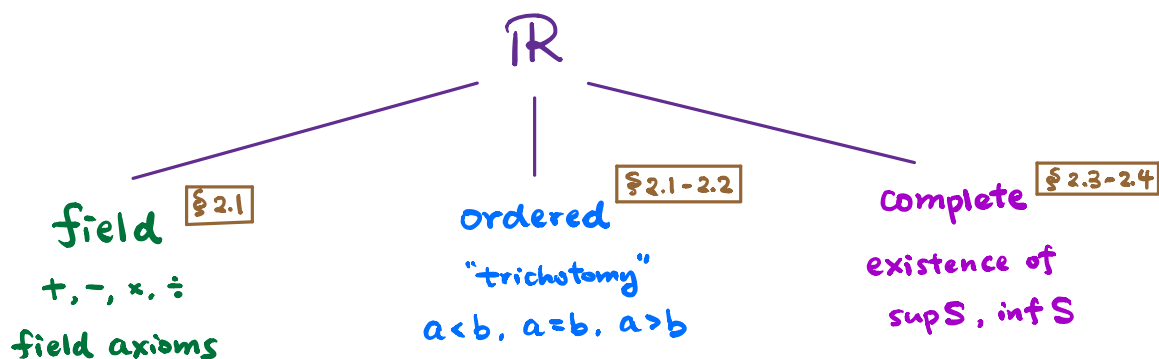
Take-home Final: May 4 (Tue) 6:00PM - May 5 (Wed) 6:00PM

Topics to be covered: (refer to Bartle (4<sup>th</sup> Ed.))

- § 2.1 - 2.5 (except binary/decimal representations)
- § 3.1 - 3.5
- § 4.1 - 4.2
- § 5.1 - 5.4 (except approximation by step functions/polynomials)

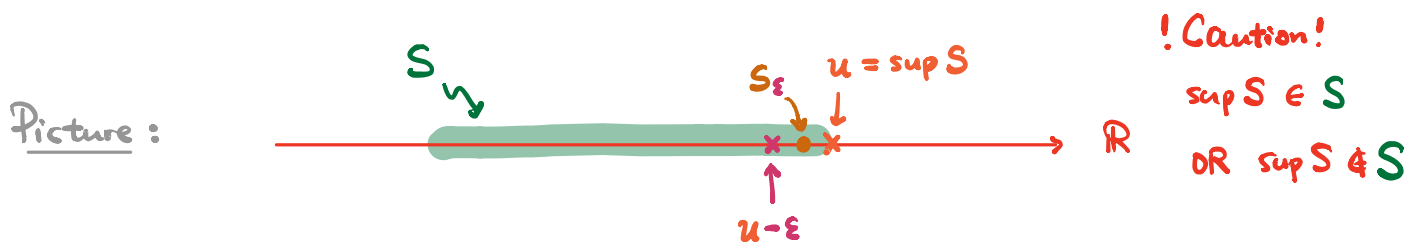
## REVIEW SESSION

### Chapter 2 The Real Numbers



Completeness Property: Every  $\emptyset \neq S \subseteq \mathbb{R}$  that is **bounded above** has a supremum in  $\mathbb{R}$ .

Def<sup>n</sup>:  $u = \sup S$   $\Leftrightarrow \begin{cases} u \geq s \quad \forall s \in S \\ \forall \epsilon > 0, \exists s_\epsilon \in S \text{ st. } u - \epsilon < s_\epsilon \end{cases}$



Useful Inequalities (§ 2.2): AM-GM ineq., (reversed) triangle ineq., Bernoulli's ineq.

Useful Facts (§ 2.4):  
•  $\mathbb{N}$  is NOT bounded above (Archimedean Property)  
(from completeness). Density of  $\mathbb{Q}$  and  $\mathbb{R} \setminus \mathbb{Q}$  in  $\mathbb{R}$   
• Existence of  $\sqrt{2}$

Intervals:

§2.5

- characterization of intervals ("Connectedness")
- Nested Interval Property: ("compactness")

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

closed and bounded intervals

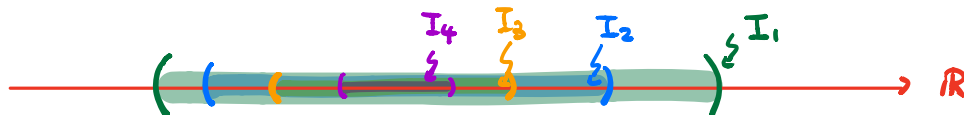
$$\Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

...

$$\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$$

$$\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$$

Picture:



## Chapter 3 Sequences (and Series)

Set

$\{x_n: n \in \mathbb{N}\} \neq$  seq.  $(x_n) = (x_1, x_2, x_3, x_4, \dots) : \mathbb{N} \rightarrow \mathbb{R}$

Def<sup>n</sup>:

§3.1  $\lim (x_n) = L \iff \forall \epsilon > 0, \exists K \in \mathbb{N}$  <sup>depends on  $\epsilon$</sup>  s.t.  $|x_n - L| < \epsilon \quad \forall n \geq K$

§3.2

Limit Thm A: If  $\lim (x_n)$  and  $\lim (y_n)$  exist, then

- $\lim (x_n \pm y_n) = \lim (x_n) \pm \lim (y_n)$
- $\lim (x_n y_n) = \lim (x_n) \lim (y_n)$   $(\frac{1}{n}) \rightarrow 0$
- $\lim \left( \frac{x_n}{y_n} \right) = \frac{\lim (x_n)}{\lim (y_n)}$  ← Provided:  $y_n \neq 0, \lim (y_n) \neq 0$

§3.2

Limit Thm B: If  $\lim (x_n)$  and  $\lim (y_n)$  exist, then

"  $x_n \leq y_n \quad \forall n \in \mathbb{N} \implies \lim (x_n) \leq \lim (y_n)$  "

[! Caution! Only get " $\leq$ " even if  $x_n < y_n \quad \forall n \in \mathbb{N}$ . E.g.  $0 < \frac{1}{n}$  ]

FACT:  $(x_n)$  convergent  $\implies (x_n)$  bounded

§3.2

← + monotone

Monotone Convergence Thm §3.3



$$(x_n) = (-1)^n$$

$$(x_n) = \left(\frac{1}{n}\right)$$

$$(x_n) = \left(\frac{(-1)^n}{n}\right)$$

To show  $(x_n)$  divergent

(I)  $(x_n)$  unbounded §3.2

(II)  $\exists$  two subseq of  $(x_n)$

$$(x_{n_k}) \rightarrow L \quad \text{§3.4}$$

$$(x_{m_k}) \rightarrow L' \quad \#$$

do NOT need to know the limit

To show  $(x_n)$  convergent

(I)  $\epsilon$ - $K$  definition §3.1

(II) Limit thms §3.2

(III) Squeeze thm §3.2

\* (IV) Monotone Convergence Thm §3.3

\* (V) Cauchy criteria §3.5

Def<sup>n</sup>: §3.5

$(x_n)$  is Cauchy  $\Leftrightarrow$

$$\forall \epsilon > 0, \exists H \in \mathbb{N} \text{ st.}$$

$$|x_n - x_m| < \epsilon \quad \forall n, m \geq H$$

depends on  $\epsilon$

no relation between them

Cauchy Criteria:  $(x_n)$  convergent  $\Leftrightarrow$   $(x_n)$  Cauchy  
"iff"

§3.4

Bolzano-Weierstrass Thm: Any bounded seq has a convergent subseq.

[! Caution! May have different subseq's converging to different limits.  
E.g.  $(-1)^n \rightsquigarrow$  limsup & liminf]

## Chapter 4 Limits (of functions)

Setup:  $f: A \rightarrow \mathbb{R}$ ,  $c \in \mathbb{R}$  is a cluster pt of  $A$

[! Caution! Either  $c \in A$  or  $c \notin A$  is possible E.g.)  $A = [0, 1]$ ]

Def<sup>n</sup>:

$$\lim_{x \rightarrow c} f(x) = L \quad \Leftrightarrow$$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ st.}$$

$$|f(x) - L| < \epsilon \quad \forall x \in A, \quad \underbrace{0 < |x - c| < \delta}_{x \neq c}$$

depends on  $\epsilon$

Sequential Criteria: §4.1

$$\lim (f(x_n)) = L$$

limit of seq.

$$\lim_{x \rightarrow c} f(x) = L \quad \Leftrightarrow$$

$$\forall \text{ seq. } (x_n) \text{ in } A \setminus \{c\} \text{ st. } \lim (x_n) = c$$

limit of function

$x_n \neq c$

[FACT: Useful to show  $\lim_{x \rightarrow c} f(x)$  does NOT exist. E.g.)  $f(x) = \sin \frac{1}{x}$ ]

• Limit Thm A and B carries over from seq. to functions

§4.2

# Chapter 5 Continuous Functions

§5.1

Def<sup>n</sup>:  $f: A \rightarrow \mathbb{R}$

is continuous at  $c \in A$

$$\Leftrightarrow \forall \epsilon > 0, \exists \delta > 0 \text{ st. } |f(x) - f(c)| < \epsilon \quad \forall x \in A, |x - c| < \delta$$

← depends on  $\epsilon$  (and  $c$ )

no  $0 <$

[! Caution! Unlike  $\lim_{x \rightarrow c} f(x)$ , we NEED  $c \in A$  here.]

Sequential Criteria: §5.1

$f: A \rightarrow \mathbb{R}$   
is cts at a  
cluster pt.  $c \in A$

$$\Leftrightarrow \lim (f(x_n)) = f(c) \quad \forall \text{ seq. } (x_n) \text{ in } A \text{ st. } \lim(x_n) = c$$

[FACT: Useful to show Discontinuity at  $c$ . E.g.)  $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ ]

§5.2

Facts:  $f, g$  cts  $\Rightarrow f \pm g, fg, f/g, f \circ g$  (composition) new!

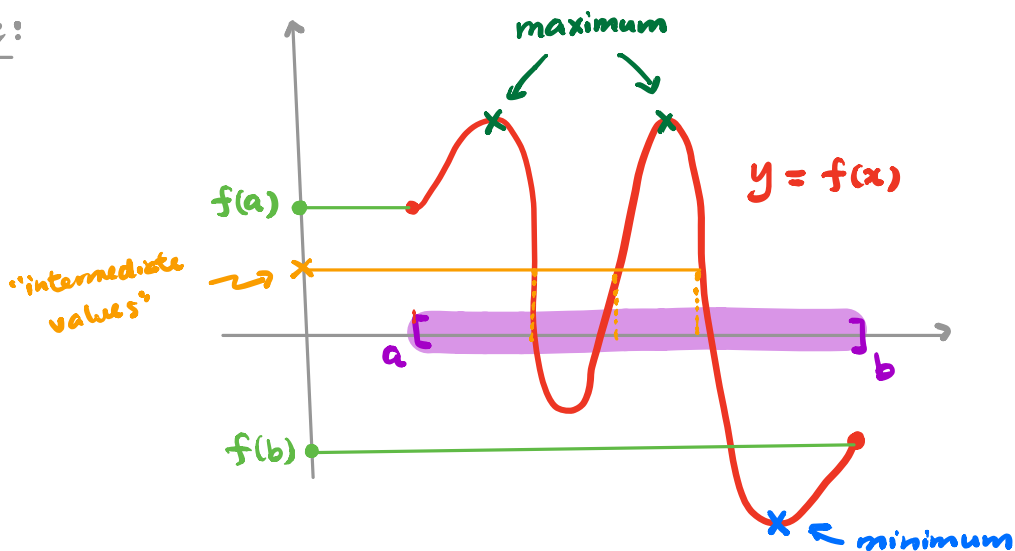
closed + bdd

Two Theorems for cts  $f: [a, b] \rightarrow \mathbb{R}$  §5.3

Extreme Value Thm:  $f$  achieves its absolute maximum and minimum.

Intermediate Value Thm:  $f$  achieves ALL intermediate values between  $f(a)$  and  $f(b)$ .

Picture:



Def<sup>2</sup>: §5.4

$f: A \rightarrow \mathbb{R}$   
is uniformly cts  
(on  $A$ )

$\Leftrightarrow$

$\forall \epsilon > 0, \exists \delta > 0$  st.  
 $|f(u) - f(v)| < \epsilon \quad \forall u, v \in A, |u - v| < \delta$

*depends ONLY on  $\epsilon$ , but NOT  $u, v$*

FACT:  $f$  unif. cts on  $A \implies f$  cts on  $A$  (i.e. at ALL  $c \in A$ )

e.g.)  $f(x) = x$

$\Leftarrow$  (with a red X)  
 $\therefore \delta$  may depend  
on  $c \in A$

e.g.  $f(x) = \frac{1}{x}$

Two Important Thm about uniform continuity §5.4

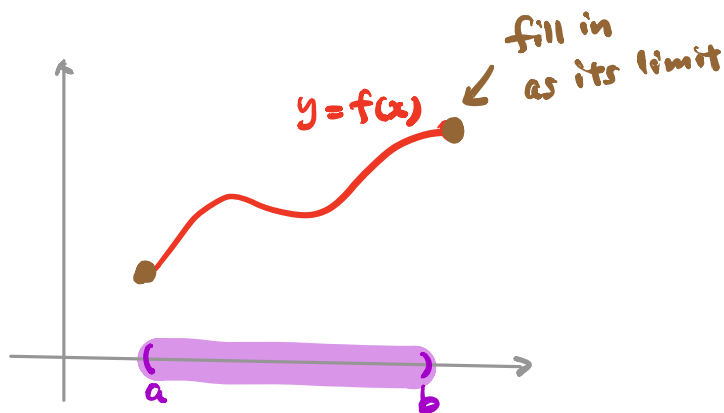
Uniform Continuity Thm:

Any cts  $f: [a, b] \rightarrow \mathbb{R}$  is uniformly cts.  
*closed + bdd*

Continuous Extension Thm:

Any uniformly cts  $f: (a, b) \rightarrow \mathbb{R}$  can be continuously extended to  $[a, b]$ .

Picture:



~ END OF REVIEW SESSION ~

Good Luck!